Value at Risk Model Risk

Carol Alexander† and Jose Maria Sarabia‡

† ICMA Centre, Henley Business School at Reading University
‡ Department of Economics, University of Cantabria, Spain

www.carolalexander.org & www.marketriskanalysis.com

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1. Motivation

- In July 2009 the Basel Committee issued *Revisions to the Basel II Market Risk Framework* – implementation due December 2010

- Section IV, *Changes to the Supervisory Review Process for Market Risk* → additional reserves to cover all sources of model risk in the internal models used to compute the market risk capital charge

- As yet, there is no generally accepted framework for quantifying model risks

- We present a totally new and rigorous framework for deriving a capital charge supplement to cover VaR model risks. With the regulator’s cooperation, it would be straightforward to implement
What is Model Risk?

- The term ‘model risk’ is frequently applied to encompass various sources of uncertainty in statistical models, but there is no general consensus on its precise definition.

- For our purposes, model risk stems from two sources: model choice and parameter estimation error, defined as follows:
  
  **Model choice**: model risk stemming from inappropriate assumptions about the form of the statistical model for the random variable. We never know the ‘true’ model, except in a simulation environment.

  **Parameter estimation error**: model risk stemming from uncertainty in the parameter values of the chosen model. We can never estimate parameter values exactly because we are constrained by a finite set of observations on the variable.
Non-Existance of a ‘True’ Model

- It is futile to propose the existence of a unique and measurable ‘true’ model – such an exercise is beyond our realm of knowledge.
- In practice, we only observe *realisations* of the data generation process of a random variable.
- As well as observations, the *state of knowledge* includes *subjective beliefs* about the process.
- Hence, no process exists that encapsulates the state of knowledge of the world, which could be labelled the unique ‘true’ process.
- Instead, an explicit statement of regulator’s or manager’s state of knowledge should be taken as the *benchmark* for assessing model risk.
Model Risk in the Real-World and Risk-Neutral Measures

1. **Valuation Model Risk**
   - Risk-neutral measure

2. **Risk Model Risk**
   - Real-world measure
2. Quantile Model Risk

- Our benchmark model will be the distribution $F$ that embodies *no more and no less* than our complete state of knowledge $K$: in other words, it is the *maximum entropy* distribution (MED) based on $K$.

- The (Shannon) entropy of a probability density function $g(x)$, $x \in \mathcal{R}$ is a measure of uncertainty (its negative is a measure of information) defined as

$$H(g) = -\mathbb{E}_g[\log g(x)] = -\int_{\mathcal{R}} g(x) \log g(x) \, dx$$

- The *maximum entropy* density maximizes $H(g)$ subject to a set of conditions on $g(x)$ imposed by $K$.

- The criterion is to be as vague as possible (i.e. to maximize uncertainty) given the constraints imposed by $K$. 
Formal Definitions

- A model is a pair \( \{ \hat{F}, \hat{K} \} \) where \( \hat{F} \) is a distribution class and \( \hat{K} \) is a filtration (i.e. \( \hat{K} \) includes the observable data and the distribution’s parameter values).

- Quantile model risk arises because \( \{ \hat{F}, \hat{K} \} \neq \{ F, K \} \) for two possible reasons:
  - \( \hat{K} \neq K \), e.g. \( \hat{K} \) may be derived from an industry standard that at least one year of observed data must be used, yet \( K \) may include the belief that only the last six months of data are actually relevant.
  - \( \hat{F} \) may not be the MED, even based on \( \hat{K} \), e.g. the execution of enterprise-wide risk models may require that \( \hat{F} \) be very simple, e.g. normal.
Quantile Probability Uncertainty

- Let $\alpha$ be a fixed probability. The $\alpha$ quantile of the distribution $F$ of a random variable $X$ is
  \[ q^F_\alpha = F^{-1}(\alpha) \] (1)

- Statistical models provide an estimate $\hat{F}$ of $F$, and to use this to estimate $q^F_\alpha$. That is, instead of (1) we use $q^{\hat{F}}_\alpha = \hat{F}^{-1}(\alpha)$, and in the presence of model risk $q^{\hat{F}}_\alpha \neq q^F_\alpha$

- So $q^{\hat{F}}_\alpha$ is at a different quantile, the $\hat{\alpha}$ quantile of $F$, i.e. $q^{\hat{F}}_\alpha = q^{\hat{F}}_{\hat{\alpha}}$

- In other words, the probability of our estimated quantile, under the MED, is not $\alpha$, it is
  \[ \hat{\alpha} = F(\hat{F}^{-1}(\alpha)) \] (2)
Illustration
Example: Distribution of $\hat{\alpha}$ due to Sampling Error

- Generate a random sample of a fixed size (e.g. 1000) from a known distribution (e.g. $N(0, 1)$) and compute
  - the $\alpha$ quantile of the sample
  - the quantile probability $\hat{\alpha}$ under the known distribution
- Repeat many times and draw the histogram of $\hat{\alpha}$
Measuring Quantile Model Risk

- We can measure quantile model risk by the deviation of $\hat{\alpha}$ from $\alpha$, i.e. the distribution of the quantile probability errors

$$e(\alpha|F, \hat{F}) = \hat{\alpha} - \alpha$$  \hspace{1cm} (3)

- If the mean error $\bar{e}(\alpha|F, \hat{F})$ is significantly different from zero the model suffers from a systematic, measurable *bias* at the $\alpha$ quantile

  - A significant and positive (negative) mean indicates a systematic over (under) estimation of the $\alpha$ quantile of the MED

- Even if the model is unbiased, it may still lack *efficiency* if the dispersion of $e(\alpha|F, \hat{F})$ is great

  - Measures of this dispersion quantify the model *uncertainty*, which is a major element of quantile model risk
3. Application to VaR Model Risk

1. Given a VaR model, are some portfolios more sensitive to its model risk than others?

2. Given a portfolio, which of the available VaR models has least model risk?

3. How can VaR model risk be included in risk capital requirements?

To illustrate how our framework can answer these questions, we consider three commonly used VaR models having varying degrees of model risk.
Experiment

- Consider an experiment in which a portfolio’s returns are simulated based on a known data generation process (the MED).
- This way we can control the degree of VaR model risk.
- Assume that our MED for the returns $X_t$ at time $t$ is $\mathcal{N}(0, \sigma_t^2)$, where $\sigma_t^2$ follows an asymmetric GARCH process.
- Hence the return $x_t$ from time $t$ to $t + 1$ and its variance $\sigma_t^2$ are simulated using:

$$\sigma_t^2 = \omega + \alpha (x_{t-1} - \lambda)^2 + \beta \sigma_{t-1}^2, \quad x_t | \mathcal{I}_t \sim \mathcal{N}(0, \sigma_t^2) \quad (4)$$

where $\omega > 0$, $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$ and $\mathcal{I}_t = (x_{t-1}, x_{t-2}, \ldots)$

- For the simulated returns the parameters of (4) are assumed to be:

$$\omega = 1.5 \times 10^{-6}, \alpha = 0.04, \lambda = 0.005, \beta = 0.95 \quad (5)$$
The Models

- Consider three models employing different processes for $\hat{\sigma}_t^2$.

- Model 1 is the correct choice of model but with incorrect parameter values: instead of (5) we employ the fitted model:

$$\hat{\sigma}_t^2 = \hat{\omega} + \hat{\alpha}(x_{t-1} - \hat{\lambda})^2 + \hat{\beta}\hat{\sigma}_{t-1}^2, \quad (6)$$

with

$$\hat{\omega} = 2 \times 10^{-6}, \hat{\alpha} = 0.0515, \hat{\lambda} = 0.01, \hat{\beta} = 0.92 \quad (7)$$

- The steady-state volatility estimate is therefore correct (25%) but since $\hat{\alpha} > \alpha$, $\hat{\beta} < \beta$ and $\hat{\lambda} > \lambda$, $\hat{\sigma}_t$ has a greater reaction but less persistence to innovations in the returns, and especially to negative returns, compared with $\sigma_t$. 
The Models

- Model 2 is a slightly inappropriate model: a simplified version of (4) with:
  \[ \hat{\omega} = \hat{\lambda} = 0, \hat{\alpha} = 0.06, \hat{\beta} = 0.94 \]  
  (8)

- This is the RiskMetrics EWMA estimator, under which a steady-state volatility is not defined

- Model 3 is the RiskMetrics ‘regulatory’ estimator, which is even more inappropriate:
  \[ \hat{\alpha} = \hat{\lambda} = \hat{\beta} = 0, \hat{\omega} = \frac{1}{250} \sum_{i=1}^{250} x_{t-i}^2 \]  
  (9)
Simulations and Estimations

- A time series of 10,000 returns $\{x_t\}_{t=1}^{10,000}$ is simulated from the model (4) with parameters (5).

- Thus, at each time $t$ we have a normal maximum entropy benchmark $F_t = F(X_t | K_t)$ for quantifying VaR model risk.

- For each of the three models, at each time $t$ we estimate the daily VaR using $\Phi^{-1}(\alpha)\hat{\sigma}_t$.

- Then we compute the probability $\hat{\alpha}_t$ associated with this quantile, under $F_t$. Because

$$\Phi^{-1}(\hat{\alpha}_t)\sigma_t = \Phi^{-1}(\alpha)\hat{\sigma}_t$$

this probability is given by

$$\hat{\alpha}_t = \Phi \left[ \Phi^{-1}(\alpha)\frac{\hat{\sigma}_t}{\sigma_t} \right]$$
Distribution of 1% Daily VaR Estimates

Here VaR is expressed as a percentage of portfolio value.
Distribution of $\hat{\alpha}$ when $\alpha = 1\%$
Sample Statistics for $\hat{\alpha}$

<table>
<thead>
<tr>
<th></th>
<th>AGARCH</th>
<th>EWMA</th>
<th>Regulatory</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10%</td>
<td>Mean</td>
<td>0.11%</td>
<td>0.16%</td>
</tr>
<tr>
<td></td>
<td>Stddev</td>
<td>0.0007</td>
<td>0.0013</td>
</tr>
<tr>
<td></td>
<td>RMSE ($\hat{\alpha} - \alpha$)</td>
<td>0.07%</td>
<td>0.14%</td>
</tr>
<tr>
<td>1%</td>
<td>Mean</td>
<td>1.03%</td>
<td>1.25%</td>
</tr>
<tr>
<td></td>
<td>Stddev</td>
<td>0.0042</td>
<td>0.0059</td>
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<tr>
<td></td>
<td>RMSE ($\hat{\alpha} - \alpha$)</td>
<td>0.42%</td>
<td>0.64%</td>
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<tr>
<td>5%</td>
<td>Mean</td>
<td>4.97%</td>
<td>5.44%</td>
</tr>
<tr>
<td></td>
<td>Stddev</td>
<td>0.0103</td>
<td>0.0123</td>
</tr>
<tr>
<td></td>
<td>RMSE ($\hat{\alpha} - \alpha$)</td>
<td>1.03%</td>
<td>1.31%</td>
</tr>
</tbody>
</table>
4. Adjustments to Capital Reserves

- Regard $\hat{\alpha}$ as a random variable with a distribution that is generated by our two sources of model risk, i.e. model choice and parameter estimation error.
- Because $\hat{\alpha}$ is a probability it has range $[0, 1]$ and we may assume that $\hat{\alpha} \sim \mathcal{GB}(a, p, q)$.
- Then the $\alpha$ quantile of our model, $F^{-1}(\alpha) = F^{-1}(\hat{\alpha})$, adjusted for model risk, also becomes a random variable – we write:

$$Q(\alpha|F, \hat{F}) = F^{-1}(\hat{\alpha}), \quad \hat{\alpha} \sim \mathcal{GB}(a, p, q)$$

- Now the model-risk-adjusted quantile $Q(\alpha|F, \hat{F})$ has a GBG distribution with parent $F$ (the MED).
- Denoting by $f$ the density of $F$, the density of $Q(\alpha|F, \hat{F})$ is

$$g_F(v; a, p, q) = B(p, q)^{-1} f(v) [a F(v)^{ap-1} (1 - F(v)^a)^{q-1}], \quad v \in \mathcal{R}$$
Beta fits to $\hat{\alpha}$ for $\alpha = (a) \ 0.10\%, \ (b) \ 1\% \ and \ (c) \ 5\%$
Distributions of $-Q(\alpha|F, \hat{F})$ at $\alpha = 1\%$

Model-risk-adjust VaR again expressed as % of portfolio value.
Bias Adjustment

- The bias of the model is measured by $E[Q(\alpha|F, \hat{F})]$.

- Financial regulators are interested in the lower tail (i.e. small $\alpha$).

- Value-at-Risk (i.e. the negative of the quantile) is systematically underestimated iff

$$E[-Q(\alpha|F, \hat{F})] < -q^F_{\alpha}, \text{ i.e. } E[Q(\alpha|F, \hat{F})] > q^F_{\alpha}$$

- Any bias can be removed by adding to $q^F_{\alpha}$ the difference

$$q^F_{\alpha} - E[Q(\alpha|F, \hat{F})]$$

- The bias-adjusted quantile then has expectation $q^F_{\alpha}$.
Uncertainty Buffer

- The more dispersed the distribution of $Q(\alpha | F, \hat{F})$, the greater the potential for $q_{\alpha}^{\hat{F}}$ to deviate from $q_{\alpha}^{F}$.

- Risk estimates are conservative. Hence, we add to the bias-adjusted quantile an *uncertainty buffer* equal to

$$E[Q(\alpha | F, \hat{F})] - G_{F}^{-1}(y)$$

where $G_{F}$ is the distribution function of $Q(\alpha | F, \hat{F})$.

- This way, we are $(1 - y)\%$ confident that the bias and uncertainty adjusted quantile is no less than $q_{\alpha}^{F}$.

- The confidence level here is a matter for subjective choice – it determines the penalty imposed for model risk.
Model-risk-adjusted $\alpha$ Quantile

- After the bias and uncertainty adjustment we obtain:

$$q_{\alpha}^{\hat{F}} + \{q_{\alpha}^{F} - E[Q(\alpha|F, \hat{F})]\} + \{E[Q(\alpha|F, \hat{F})] - G_{F}^{-1}(y)\} = q_{\alpha}^{\hat{F}} + q_{\alpha}^{F} - G_{F}^{-1}(y)$$

- The computation of $E[Q(\alpha|F, \hat{F})]$ can be circumvented except when the decomposition into bias and uncertainty components is required.

- The mean $E[Q(\alpha|F, \hat{F})]$ and quantiles $G_{F}^{-1}(y)$ of GBG distributions have simple expansion approximations that are generally quite accurate.
Implementing Adjustments – Simulation Experiment

- Returning to our simulation results we pick a random point in time when the simulated process volatility was at 25%

- At this point, we obtained the following values for VaR under the simulated (maximum entropy) process:

  \[-q_{0.1\%}^F = 4.886\%, \quad -q_{1\%}^F = 3.678\% \quad \text{and} \quad -q_{5\%}^F = 2.601\%\]

- Then, for each model and each \(\alpha\), we compute the model-risk-adjusted means \((\text{i.e. } E[Q(\alpha|F, \hat{F})])\) and hence the bias adjustment \(q_{\alpha}^F - E[Q(\alpha|F, \hat{F})]\)

- Then we compute \(G_F^{-1}(5\%)\) and hence the 95% uncertainty buffer \(E[Q(\alpha|F, \hat{F})] - G_F^{-1}(0.05)\).
Summary Statistics for $-Q(\alpha|F, \hat{F})$, $\alpha = 0.1\%, 1\%, 5\%$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>AGARCH</th>
<th>EWMA</th>
<th>Regulatory</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1%</td>
<td>Mean</td>
<td>4.919%</td>
<td>4.793%</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>-0.033%</td>
<td>0.093%</td>
</tr>
<tr>
<td></td>
<td>Quantile</td>
<td>4.447%</td>
<td>4.177%</td>
</tr>
<tr>
<td></td>
<td>95% Uncert. Buffer</td>
<td>0.472%</td>
<td>0.615%</td>
</tr>
<tr>
<td>1%</td>
<td>Mean</td>
<td>3.703%</td>
<td>3.608%</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>-0.025%</td>
<td>0.070%</td>
</tr>
<tr>
<td></td>
<td>Quantile</td>
<td>3.348%</td>
<td>3.145%</td>
</tr>
<tr>
<td></td>
<td>95% Uncert. Buffer</td>
<td>0.355%</td>
<td>0.463%</td>
</tr>
<tr>
<td>5%</td>
<td>Mean</td>
<td>2.618%</td>
<td>2.551%</td>
</tr>
<tr>
<td></td>
<td>Bias</td>
<td>-0.017%</td>
<td>0.050%</td>
</tr>
<tr>
<td></td>
<td>Quantile</td>
<td>2.366%</td>
<td>2.224%</td>
</tr>
<tr>
<td></td>
<td>95% Uncert. Buffer</td>
<td>0.252%</td>
<td>0.327%</td>
</tr>
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Motivation Quantile Model Risk VaR Model Risk Adjustments to Capital Reserves

RaVaR: Risk Adjusted VaR

- At the time of the model risk adjustment, the model’s estimated volatilities are shown. Then, for each quantile, VaR is based on a normal distribution with this volatility.

<table>
<thead>
<tr>
<th>α</th>
<th>Volatility</th>
<th>AGARCH</th>
<th>EWMA</th>
<th>Regulatory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.10%</td>
<td>VaR</td>
<td>5.277%</td>
<td>4.678%</td>
<td>5.509%</td>
</tr>
<tr>
<td></td>
<td>Bias Adj. VaR</td>
<td>5.244%</td>
<td>4.772%</td>
<td>5.434%</td>
</tr>
<tr>
<td></td>
<td>95% RaVaR</td>
<td>5.716%</td>
<td>5.387%</td>
<td>6.483%</td>
</tr>
<tr>
<td>1%</td>
<td>VaR</td>
<td>3.972%</td>
<td>3.522%</td>
<td>4.147%</td>
</tr>
<tr>
<td></td>
<td>Bias Adj. VaR</td>
<td>3.948%</td>
<td>3.592%</td>
<td>4.091%</td>
</tr>
<tr>
<td></td>
<td>95% RaVaR</td>
<td>4.303%</td>
<td>4.055%</td>
<td>4.880%</td>
</tr>
<tr>
<td>5%</td>
<td>VaR</td>
<td>2.809%</td>
<td>2.490%</td>
<td>2.932%</td>
</tr>
<tr>
<td></td>
<td>Bias Adj. VaR</td>
<td>2.791%</td>
<td>2.540%</td>
<td>2.892%</td>
</tr>
<tr>
<td></td>
<td>95% RaVaR</td>
<td>3.043%</td>
<td>2.867%</td>
<td>3.451%</td>
</tr>
</tbody>
</table>
Effect on Risk Capital

- Risk capital is a multiple of VaR. Hence, the increase when replacing VaR by RaVaR at the \((1 - y)\)% confidence level is:

\[
\% \text{ risk capital increase} = \frac{\text{MeVaR} - G_F^{-1}(y)}{\text{VaR}}
\]

- In our simulations results, the final table shows the percentage increase in risk capital for different RaVaR confidence levels:

<table>
<thead>
<tr>
<th>(1 - y)</th>
<th>AGARCH</th>
<th>EWMA</th>
<th>Regulatory</th>
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<tr>
<td>95%</td>
<td>8.33%</td>
<td>15.15%</td>
<td>17.68%</td>
</tr>
<tr>
<td>90%</td>
<td>6.05%</td>
<td>12.62%</td>
<td>14.16%</td>
</tr>
<tr>
<td>85%</td>
<td>4.47%</td>
<td>10.99%</td>
<td>11.37%</td>
</tr>
<tr>
<td>80%</td>
<td>3.19%</td>
<td>9.57%</td>
<td>8.81%</td>
</tr>
<tr>
<td>75%</td>
<td>2.25%</td>
<td>8.38%</td>
<td>6.68%</td>
</tr>
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